Control Systems : Set 9 : Loopshaping (5) - Solutions

Prob 1 | For a system with open-loop transfer function

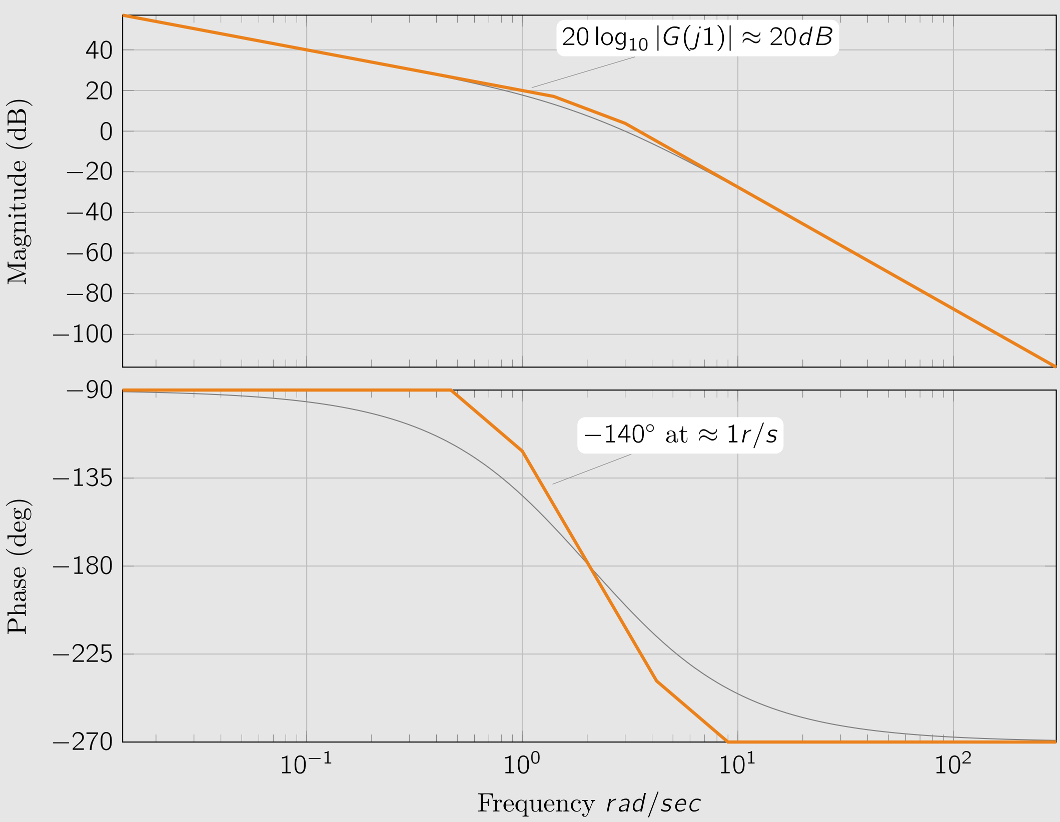
$$G(s) = \frac{10}{s(s/1.4+1)(s/3+1)}$$

design a lag compensator with unity DC gain so that $PM \geq 30°$. What is the approximate bandwidth of this system?
Note: A lag compensator with unity DC gain takes the form

$$D(s) = \frac{T_I s + 1}{\alpha T_I s + 1}, \ \alpha > 1$$

The DC gain here will be one, and the gain at high frequency will be $\frac{1}{\alpha}$.

The bode sketch is shown below



We see that the phase is approximately $-140°$ at a frequency of about $1r/s$, where the gain is about $20dB$. This would give us around $40°$ phase margin, which provides a $10°$ buffer for the phase drop resulting from the lag compensator. Therefore, we choose α to reduce the gain by $20dB$, which will cause the crossover frequency to be around $1r/s$, and therefore a phase margin of $40°$.

$$\frac{1}{\alpha} = -20dB \qquad\qquad \rightarrow \qquad\qquad \alpha = 10$$

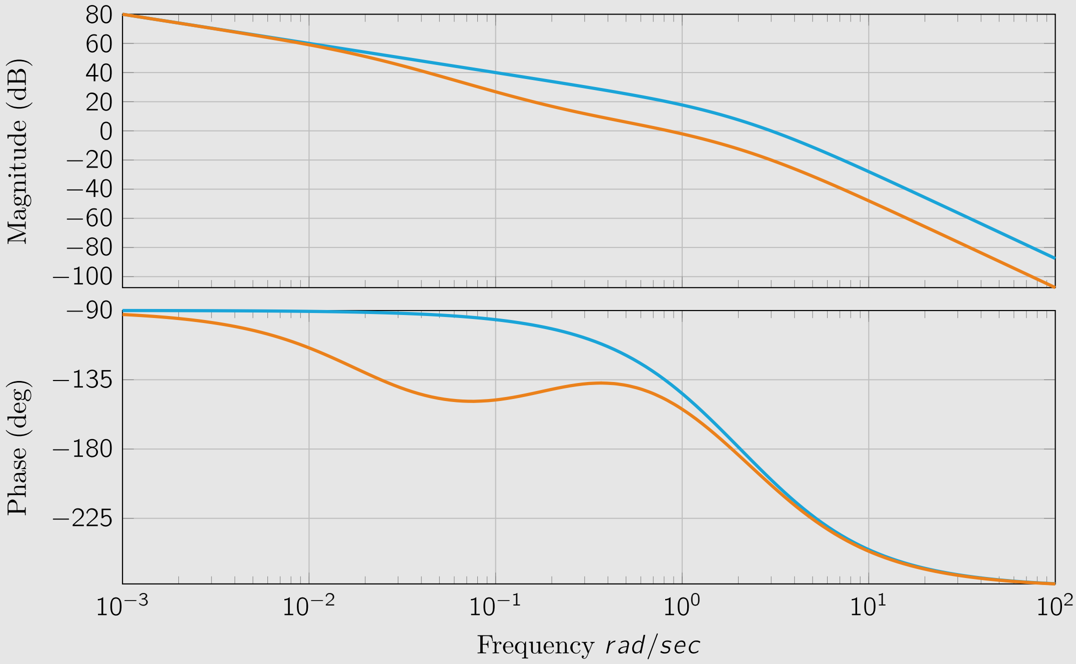
Choose T_I so that the phase drop from the lag filter does not impact the phase at the crossover frequency. We choose this about $5x$ less than the crossover frequency:

$$\frac{1}{T_I} \approx \frac{\omega_c}{5} \qquad\qquad \Rightarrow \qquad\qquad T_I = 5$$

The resulting controller is

$$D_c(s) = \frac{5s+1}{50s+1}$$

We see the bode plot of the original (blue) and compensated (orange) system below, where we note that we have a phase margin of $31°$ and a crossover frequency of $0.85 rad/sec$.



Prob 2 |

$$G(s) = \frac{100000}{s(20+s)(200+s)}$$

Design a PID controller for $G(s)$ using loopshaping to satisfy the criteria

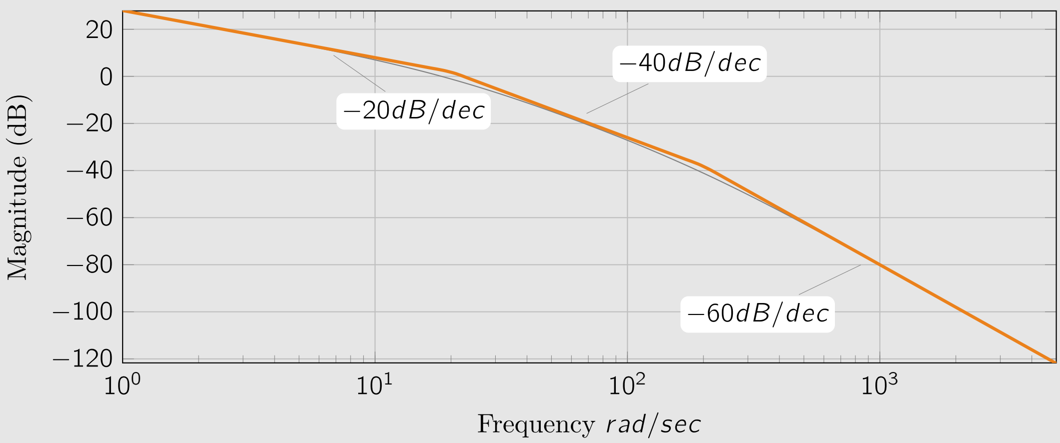
- Zero tracking for ramp inputs
- Phase margin of around 60°
- Minimum possible rise time

Hint: This is a minimum phase, stable system.

The system is minimum phase and open-loop stable, and so we can use the Bode gain-phase relationship to simplify our design process.

- Zero tracking for ramp inputs → Type 2 system required
- Phase margin of around 60° → Slope of $-20dB/dec$ at crossover
- Minimum possible rise time → Maximize bandwidth

We first sketch the magnitude Bode plot

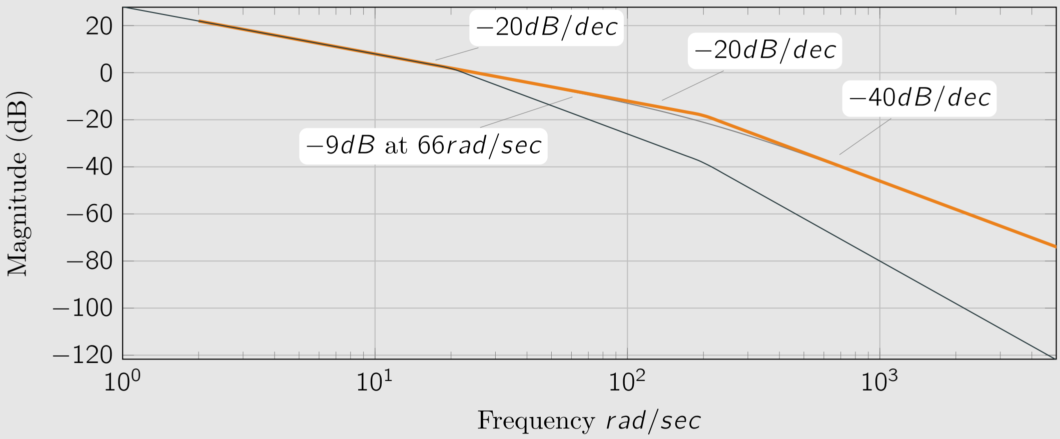


We will use a PID controller in series form, as we're doing loopshaping

$$D(s) = K \cdot (T_D s + 1) \cdot (1 + \frac{1}{T_I s})$$

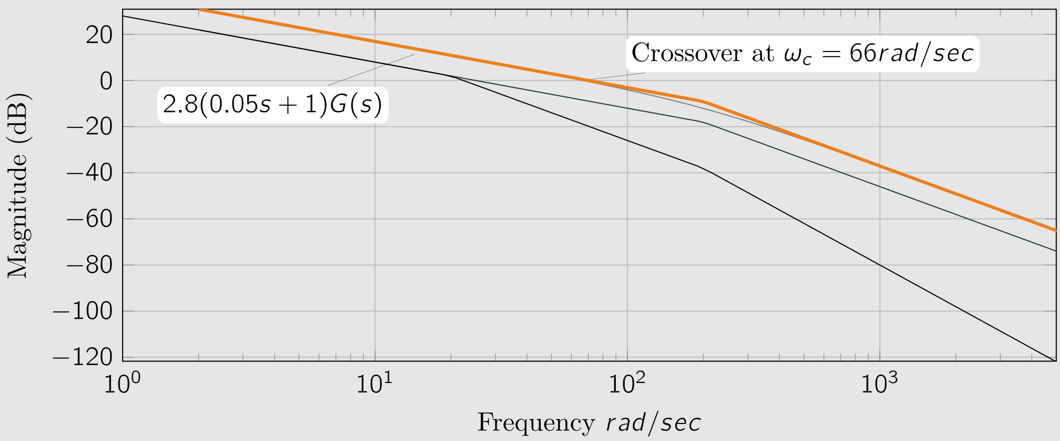
From the plot, we can see that the slope at high frequency is $-60dB/dec$. As we only have one zero to add (the derivative term), we can at most increase the slope by $20dB/dec$, meaning that at frequencies higher than $200r/s$ we cannot achieve a slope better than $-40dB/dec$. Therefore, the maximum crossover frequency achievable is below $200r/s$.
We place the controller zero at $1/T_D = 20r/s$, which will cancel the system pole (allowed because it is stable), and increase the slope at higher frequencies. Specifically between the

$20r/s$ and $200r/s$ we will have a slope of about $-20dB/dec$, meaning that this is where we want the crossover frequency. This first step is shown in the bode plot below.



We can now choose the system gain to shift the plot up and choose the largest possible bandwidth. We need the slope at crossover to be $-20dB/dec$ for at least a decade. Therefore, we place the crossover a 1/2 decade below $200r/s$, or at $200/3 = 66r/s$. The gain of the system at $66r/s$ is $-9dB$, so we choose $K = 9dB = 2.8$. The result of the gain and the derivative term are shown in the bode plot below.
Note: If you want a decade around a given frequency x, you can multiply and divide by 3 to get it:

$$\log_{10}(3x) - \log_{10}(x/3) = 2\log_{10}(3) = 0.95$$



The last step is to decrease the slope at lost frequencies to $-40dB/dec$ so that the system will track ramp inputs without any steady-state error, which is done by adding the integrator. We want the break frequency of the integrator to be as high as possible, to give a faster response, but low enough that it does not impact the phase margin at crossover. We therefore place it a half decade below our crossover frequency at

$$\frac{1}{T_i} = \frac{\omega_c}{3} \approx 20$$

Our PID controller is therefore

$$D(s) = 2.8\left(\frac{s}{20}+1\right)\left(1+\frac{20}{s}\right)$$

The final compensated bode plot is shown below, along with the closed-loop step response.